

# Gauge Theories in Noncommutative Homogeneous Kähler Manifolds

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## Abstract

We construct a gauge theory on a noncommutative homogeneous Kähler manifold, where we employ the deformation quantization with separation of variables for Kähler manifolds formulated by Karabegov. A key point in this construction is to obtaining vector fields which act as inner derivations for the deformation quantization. We show that these vector fields are the only Killing vector fields. We give an explicit construction of this gauge theory on noncommutative  $\mathbb{C}P^N$  and noncommutative  $\mathbb{C}H^N$ .

## 1 Introduction

Field theories on noncommutative spaces appear in various phenomena in physics. For example, effective theories on D-branes with NS-NS B field backgrounds give rise to gauge theories on noncommutative spaces [36]. As another example, in matrix models [2, 17], noncommutative field theories corresponding to fuzzy spaces appear when one expands the models around some classical solutions.

A typical noncommutative space is the noncommutative  $\mathbb{R}^d$ . Field theories on the noncommutative  $\mathbb{R}^d$  have many intriguing properties. For example, there is work on the existence of noncommutative instantons [31], noncommutative scalar solitons [14], etc. as classical solutions and the appearance of UV-IR mixing [28] at the quantum level (see also the review papers [8, 33, 37], for examples). It is important

to investigate whether field theories on more generic noncommutative manifolds have similar properties. However, field theories on noncommutative manifolds are not well understood at present, except for a few examples such as, noncommutative tori,  $S^2$ , etc.

Several methods to construct noncommutative manifolds have been proposed, including the important approach by the deformation quantization. Deformation quantization was first introduced in [3]. After [3], several alternative methods of deformation quantization were proposed [9, 32, 12, 26]. In particular, deformation quantization of Kähler manifolds was studied in [29, 30, 5, 6]. We study gauge field theories on noncommutative Kähler manifolds based on the deformation quantization with separation of variables introduced by Karabegov to quantize Kähler manifolds [18, 19, 21].

The purpose of this paper is to construct gauge theories on noncommutative homogeneous Kähler manifolds. Field theories need to define differentials on base spaces. Note that the usual differentiations by coordinates in a noncommutative space may not be derivations; in other words, they do not satisfy the Leibniz rule for star products in general. We use inner derivations as differentials, which are defined by commutators with a function  $P$  under a star product, *i.e.*  $[P, \cdot]_*$ . These operators automatically satisfy the Leibniz rule. For a generic  $P$ , the inner derivation  $[P, \cdot]_*$  includes higher derivative terms. We investigate conditions on  $P$  such that the inner derivation includes no higher derivative terms, and show that for Kähler manifolds, a necessary and sufficient condition is that  $P$  is the Killing potential. For homogeneous Kähler manifolds  $\mathcal{G}/\mathcal{H}$ , there are Killing vectors  $\mathcal{L}_a$  which constitute the Lie algebra of the isometry group  $\mathcal{G}$ . The Killing potential  $P_a$  corresponding to  $\mathcal{L}_a$  exists, and  $\mathcal{L}_a$  is represented by the inner derivation  $\mathcal{L}_a = \{P_a, \cdot\} = -\frac{i}{\hbar}[P_a, \cdot]_*$ .

Using these Killing potentials, we construct a gauge theory on noncommutative homogeneous Kähler manifolds. In our previous papers [34, 35], we studied deformation quantizations with separation of variables for  $\mathbb{C}P^N$  and  $\mathbb{C}H^N$ , and gave explicit expressions for the star products. Using these results, we describe  $U(n)$  gauge theories on noncommutative  $\mathbb{C}P^N$  and on noncommutative  $\mathbb{C}H^N$ , as examples.<sup>1</sup>

The organization of this article is as follows. In section 2, after we review deformation quantization with separation of variables for Kähler manifolds proposed by Karabegov, we study differentials on noncommutative Kähler manifolds. The conditions under which inner derivations become vector fields (Killing vector fields) are provided. We then construct gauge theories on noncommutative homogeneous Kähler manifolds. In section 3, we discuss gauge theories on noncommutative  $\mathbb{C}P^N$

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<sup>1</sup>On other types of noncommutative  $\mathbb{C}P^N$ , different gauge theories have been constructed. For example, a gauge theory on fuzzy  $\mathbb{C}P^N$  is studied in [7, 15].

and  $\mathbb{C}H^N$ , as concrete examples. In section 4, we summarize our results and give some further discussion.

## 2 Deformation quantization of gauge theories with separation of variables

### 2.1 Deformation quantization with separation of variables

We briefly review the deformation quantization with separation of variables for Kähler manifolds, which proposed by Karabegov [19].

Let  $\Phi$  be a Kähler potential and  $\omega$  a Kähler 2-form for  $N$ -dimensional Kähler manifolds  $M$ :

$$\omega := ig_{k\bar{l}}dz^k \wedge d\bar{z}^l, \quad g_{k\bar{l}} := \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l}. \quad (2.1)$$

We denote the inverse of the metric  $(g_{k\bar{l}})$  as  $(g^{\bar{k}l})$ . We use the following abbreviations

$$\partial_k = \frac{\partial}{\partial z^k}, \quad \partial_{\bar{k}} = \frac{\partial}{\partial \bar{z}^k}. \quad (2.2)$$

Deformation quantization is defined as follows. Let  $\mathcal{F}$  be a set of formal power series in  $\hbar$  with coefficients of  $C^\infty$  functions on  $M$

$$\mathcal{F} := \left\{ f \mid f = \sum_k \hbar^k f_k, \ f_k \in C^\infty(M) \right\}, \quad (2.3)$$

where  $\hbar$  is a noncommutative parameter. A star product is defined on  $\mathcal{F}$  by

$$f * g = \sum_k \hbar^k C_k(f, g), \quad (2.4)$$

such that the product satisfies the following conditions.

1.  $*$  is associative product.
2.  $C_k$  is a bidifferential operator.
3.  $C_0$  and  $C_1$  are defined as

$$C_0(f, g) = fg, \quad (2.5)$$

$$C_1(f, g) - C_1(g, f) = i\{f, g\}, \quad (2.6)$$

where  $\{f, g\}$  is the Poisson bracket.

$$4. \quad f * 1 = 1 * f = f.$$

Moreover,  $*$  is called a star product with separation of variables when it satisfies

$$a * f = af, \quad f * b = fb, \quad (2.7)$$

for any holomorphic function  $a$  and any anti-holomorphic function  $b$ . Karabegov constructed a star product with separation of variables for Kähler manifolds in terms of differential operators [18, 19], as briefly explained below. For the left star multiplication by  $f \in \mathcal{F}$ , there exists a differential operator  $L_f$  such that

$$L_f g = f * g. \quad (2.8)$$

$L_f$  is given as a formal power series in  $\hbar$

$$L_f = \sum_{n=0}^{\infty} \hbar^n A^{(n)}, \quad (2.9)$$

where  $A^{(n)}$  is a differential operator which contains only partial derivatives by  $z^i$  and has the following form

$$A^{(n)} = \sum_{k \geq 0} a_{i_1 \dots i_k}^{(n; k)} D^{\bar{i}_1} \dots D^{\bar{i}_k}, \quad (2.10)$$

where

$$D^{\bar{i}} = g^{\bar{i}j} \partial_j. \quad (2.11)$$

In particular,  $a^{(n; 0)}$  which is a  $C^\infty$  function on  $M$  acts as a multiplication operator. Note that the differential operators  $D^{\bar{i}}$  satisfy the following relations,

$$[D^{\bar{i}}, D^{\bar{j}}] = 0, \quad (2.12)$$

$$[D^{\bar{i}}, \partial_{\bar{j}} \Phi] = \delta_{ij}. \quad (2.13)$$

Karabegov showed the following theorem.

**Theorem 2.1** ([18, 19]).  *$L_f$  is uniquely determined by requiring the following conditions,*

$$L_f 1 = f * 1 = f, \quad (2.14)$$

$$[L_f, \partial_{\bar{i}} \Phi + \hbar \partial_{\bar{i}}] = 0. \quad (2.15)$$

Substituting the expression of  $L_f$  in (2.9) to the conditions (2.14) and (2.15), one obtains the following recursion relations,

$$A^{(0)} = f, \quad A^{(r)} 1 = 0, \quad (2.16)$$

$$[A^{(r)}, \partial_{\bar{i}} \Phi] = [\partial_{\bar{i}}, A^{(r-1)}], \quad (2.17)$$

for  $r \geq 1$ . In the case of  $r = 1$ , one can easily find

$$A^{(1)} = \partial_{\bar{i}} f D^{\bar{i}}, \quad (2.18)$$

where (2.13) is used. Let us observe that  $a^{(r;0)} = a_{\bar{i}}^{(r;1)} = 0$  for  $r \geq 2$  in the expressions (2.10), namely,

$$A^{(r)} = \sum_{k \geq 2} a_{\bar{i}_1 \dots \bar{i}_k}^{(r;k)} D^{\bar{i}_1} \dots D^{\bar{i}_k}, \quad r \geq 2. \quad (2.19)$$

From the condition (2.16),  $a^{(r;0)} = 0$  ( $r \geq 1$ ) trivially obeys. We then define the twisted symbol of  $A^{(r)}$  as  $a^{(r)}(\xi) = \sum a_{\bar{i}_1 \dots \bar{i}_n}^{(r;k)} \xi^{\bar{i}_1} \dots \xi^{\bar{i}_n}$ . The twisted symbol of the left hand side in (2.17) is  $\partial a^{(r)}(\xi) / \partial \xi^{\bar{i}}$  from (2.13). That of the right hand side in (2.17) does not contain the zeroth order term of  $\xi$ , because of  $a^{(r;0)} = 0$  for  $r \geq 1$ . Therefore,  $a^{(r)}$  ( $r \geq 2$ ) does not contain the first order term of  $\xi$ . This prove the assertion.

Here is a useful theorem given by Karabegov.

**Theorem 2.2** ([18, 19]). *The differential operator  $L_f$  for an arbitrary function  $f$  is obtained from the operator  $L_{\bar{z}^i}$ , which corresponds to the left  $*$  multiplication of  $\bar{z}^i$ ,*

$$L_f = \sum_{\alpha} \frac{1}{\alpha!} \left( \frac{\partial}{\partial \bar{z}} \right)^{\alpha} f(L_{\bar{z}} - \bar{z})^{\alpha}, \quad (2.20)$$

where  $\alpha$  is a multi-index.

Similarly, the differential operator  $R_f = \sum_{n=0}^{\infty} \hbar^n B^{(n)}$  corresponding to the right  $*$  multiplication by a function  $f$  contains only partial derivatives by  $\bar{z}^i$  and is determined by the conditions

$$R_f 1 = 1 * f = f, \quad (2.21)$$

$$[R_f, \partial_{\bar{i}} \Phi + \hbar \partial_{\bar{i}}] = 0. \quad (2.22)$$

$B^{(n)}$  has the following form,

$$B^{(0)} = f, \quad B^{(1)} = \partial_{\bar{i}} f D^{\bar{i}}, \quad B^{(r)} = \sum_{k \geq 2} b_{\bar{i}_1 \dots \bar{i}_k}^{(r;k)} D^{\bar{i}_1} \dots D^{\bar{i}_k}, \quad (2.23)$$

where  $D^i = g^{i\bar{j}} \partial_{\bar{j}}$ . The differential operator  $R_f$  for an arbitrary function  $f$  is obtained from the operator  $R_{z^i}$ , which corresponds to the right  $*$  multiplication by  $z^i$ ,

$$R_f = \sum_{\alpha} \frac{1}{\alpha!} \left( \frac{\partial}{\partial z} \right)^{\alpha} f(R_z - z)^{\alpha}. \quad (2.24)$$

## 2.2 Derivations in deformation quantization

A differential calculus on noncommutative spaces can be constructed based on the derivations of the algebra  $C^{\infty}(M)[[\hbar]]$  with its star product, whose derivation  $\mathbf{d}$  are linear operators satisfying the Leibniz rule, i.e.  $\mathbf{d}(f * g) = \mathbf{d}f * g + f * \mathbf{d}g$ . In commutative space, vector fields are obviously derivations. However first order differential operators in noncommutative space do not satisfy the Leibniz rule in general as we study in this section. In short, the reason is that the star product contains complicated coefficients of functions as saw in Section 2.1. In this subsection, we study inner derivations  $\mathcal{L}$ , in particular, let  $\mathcal{L}$  be a linear differential operator such that  $\mathcal{L}(f) = [P, f]_* := P * f - f * P$ , ( $P, f \in C^{\infty}(M)[[\hbar]]$ ).

Note that inner derivations are not first order differential operator, since the explicit expression of the star product  $[P, f]_*$  includes higher derivative terms of  $f$  for a generic  $P$ . In particular, inner derivations corresponding to vector fields play an important role, when we construct field theories on noncommutative spaces. In this section, we will show the following theorem.

**Theorem 2.3.** *Let  $M$  be a Kähler manifold with the  $*$  product with separation of variables given in section 2.1. Let  $P \in C^{\infty}(M)[[\hbar]]$ ,  $f$  be an arbitrary  $C^{\infty}$  function on  $M$  and  $[P, f] = P * f - f * P$  i.e. the inner derivation of the  $*$ -product mentioned above. Then  $[P, f]_* = i\hbar\{P, f\}$  if and only if  $D^i D^j P = 0$  and  $\bar{D}^i \bar{D}^j P$  for all  $i, j = 1, 2, \dots, N$ . Namely, higher derivative terms of  $f$  in  $[P, f]_*$  vanish and this inner derivation is given by some vector field when these conditions are satisfied.*

*Proof.* From the formulas (2.20) and (2.24), we find

$$\begin{aligned} [P, f]_* &= R_f P - L_f P \\ &= \sum_{\alpha} \left[ \frac{1}{\alpha!} (R_z - z)^{\alpha} P \cdot \left( \frac{\partial}{\partial z} \right)^{\alpha} - (L_{\bar{z}} - \bar{z})^{\alpha} P \cdot \left( \frac{\partial}{\partial \bar{z}} \right)^{\alpha} \right] f. \end{aligned} \quad (2.25)$$

The differential operators  $L_{\bar{z}^i}$  and  $R_{z^i}$  have the following forms,

$$L_{\bar{z}^i} = \bar{z}^i + \sum_{n=1}^{\infty} \hbar^n A_i^{(n)}, \quad (2.26)$$

$$R_{z^i} = z^i + \sum_{n=1}^{\infty} \hbar^n B_i^{(n)}. \quad (2.27)$$

From (2.18), (2.19) and (2.23),  $A_{\bar{i}}^{(n)}$  and  $B_i^{(n)}$  are given as

$$A_{\bar{i}}^{(1)} = D^{\bar{i}}, \quad A_{\bar{i}}^{(r)} = \sum_{k \geq 2} a_{i;\bar{j}_1 \dots \bar{j}_k}^{(r;k)} D^{\bar{j}_1} \dots D^{\bar{j}_k}, \quad r \geq 2, \quad (2.28)$$

$$B_i^{(1)} = D^i, \quad B_i^{(r)} = \sum_{k \geq 2} b_{i;\bar{j}_1 \dots \bar{j}_k}^{(r;k)} D^{j_1} \dots D^{j_k}, \quad r \geq 2. \quad (2.29)$$

The first order terms in  $\hbar$  in the right hand side of (2.25) give the Poisson bracket  $i\hbar\{P, f\}$ . Looking at  $(L_{\bar{z}^{i_1}} - \bar{z}^{i_1}) \dots (L_{\bar{z}^{i_k}} - \bar{z}^{i_k}) P$  for  $k \geq 2$  and  $P = \sum_{n=0}^{\infty} \hbar^n P^{(n)}$ , we have

$$(L_{\bar{z}^{i_1}} - \bar{z}^{i_1}) \dots (L_{\bar{z}^{i_k}} - \bar{z}^{i_k}) P = \sum_{n=0}^{\infty} \sum_{m_1=1}^{\infty} \dots \sum_{m_k=1}^{\infty} \hbar^{n+m_1+\dots+m_k} A_{\bar{i}_1}^{(m_1)} \dots A_{\bar{i}_k}^{(m_k)} P^{(n)}. \quad (2.30)$$

Assuming  $[P, f] = i\hbar\{P, f\}$ , namely, assuming that the all terms in (2.30) vanish, we show  $D^{\bar{i}} D^{\bar{j}} P = 0$ . The terms of the order  $\hbar^2$  in (2.30) exists only for  $k = 2$  and has the following form,

$$A_{\bar{i}_1}^{(1)} A_{\bar{i}_2}^{(1)} P^{(0)} = D^{\bar{i}_1} D^{\bar{i}_2} P^{(0)}. \quad (2.31)$$

Hence,  $D^{\bar{i}_1} D^{\bar{i}_2} P^{(0)} = 0$ , and we find

$$\sum_{m_1=1}^{\infty} \dots \sum_{m_k=1}^{\infty} \hbar^{n+m_1+\dots+m_k} A_{\bar{i}_1}^{(m_1)} \dots A_{\bar{i}_k}^{(m_k)} P^{(0)} = 0, \quad (2.32)$$

from the explicit forms of  $A_{\bar{i}}^{(r)}$ , (2.28). As the induction assumption, we set  $D^{\bar{i}} D^{\bar{j}} P^{(n)} = 0$  for  $n = 0, 1, \dots, r-1$ . Similar to the case of  $P^{(0)}$ , the following equation holds for  $n = 0, 1, \dots, r-1$ ,

$$\sum_{m_1=1}^{\infty} \dots \sum_{m_k=1}^{\infty} \hbar^{n+m_1+\dots+m_k} A_{\bar{i}_1}^{(m_1)} \dots A_{\bar{i}_k}^{(m_k)} P^{(n)} = 0, \quad (2.33)$$

and the right hand side of (2.30) becomes

$$\sum_{n=r}^{\infty} \sum_{m_1=1}^{\infty} \dots \sum_{m_k=1}^{\infty} \hbar^{n+m_1+\dots+m_k} A_{\bar{i}_1}^{(m_1)} \dots A_{\bar{i}_k}^{(m_k)} P^{(n)}. \quad (2.34)$$

The term of the order  $\mathcal{O}(\hbar^{r+2})$  in this sum exists only for  $k = 2$  and has the following form,

$$A_{\bar{i}_1}^{(1)} A_{\bar{i}_2}^{(1)} P^{(r)} = D^{\bar{i}_1} D^{\bar{i}_2} P^{(r)}. \quad (2.35)$$

Thus,  $D^{\bar{1}}D^{\bar{2}}P^{(r)} = 0$ . Therefore, it is shown that  $D^{\bar{i}}D^{\bar{j}}P = 0$  holds for all  $i, j$ .

Similarly,  $D^iD^jP = 0$  can be derived by considering  $(R_z - z)^\alpha P$ .

The converse is easily shown from the above equations.  $\square$

Real valued functions which satisfy  $D^iD^jP = 0$  and  $D^{\bar{i}}D^{\bar{j}}P = 0$  on Kähler manifolds are known as Killing potentials [13]. The Killing potential gives a holomorphic Killing vector  $\zeta^i\partial_i + \zeta^{\bar{i}}\partial_{\bar{i}} = \{P, \cdot\}$ ,

$$\zeta^i = -ig^{i\bar{j}}\partial_{\bar{j}}P = -iD^iP, \quad (2.36)$$

$$\zeta^{\bar{i}} = ig^{\bar{i}j}\partial_jP = iD^{\bar{i}}P. \quad (2.37)$$

$\zeta^i$  is holomorphic, and  $\zeta^{\bar{i}}$  is anti-holomorphic. The metric and the complex structure of the Kähler manifold are invariant under the transformations generated by the holomorphic Killing vectors,  $\delta_\zeta z^i = \zeta^i$ ,  $\delta_\zeta \bar{z}^i = \zeta^{\bar{i}}$ . Summarizing these facts, we have the following corollary

**Corollary 2.4.** *In deformation quantization defined in Section 2.1, the inner derivations given as vector fields are the Killing vector fields  $\mathcal{L}_a = \zeta_a^i\partial_i + \zeta_a^{\bar{i}}\partial_{\bar{i}}$ .*

## 2.3 Deformed gauge theory

In the previous section, we studied inner derivations given as vector fields on non-commutative Kähler manifolds. Using this, we investigate gauge theories with a gauge group  $G$  on noncommutative homogeneous Kähler manifolds  $M$  given by the deformation quantization in Section 2.1.<sup>2</sup> In the following, we consider  $U(n)$  gauge theories for simplicity. All results in this section can be applied for any matrix groups.

At first, we introduce a noncommutative  $U(n)$  transformations as a deformation of the unitary transformations. If  $g \in U(n)$ , then  $g^\dagger g = I$ , where  $g^\dagger$  is the hermitian conjugate of  $g$  and  $I$  is the identity matrix. As a natural extension, we define  $G := C^\infty(M)[[\hbar]] \otimes GL(n; \mathbb{C})$  such that for  $U = \sum_{k=0}^{\infty} \hbar^k U^{(k)}$  and  $U^\dagger = \sum_{k=0}^{\infty} \hbar^k U^{(k)\dagger} \in G$ ,

$$U^\dagger * U = \sum_{n=0}^{\infty} \hbar^n \sum_{m=0}^n U^{(m)\dagger} * U^{(n-m)} = I. \quad (2.38)$$

This condition is imposed for each order of  $\hbar$ . For arbitrary  $U^{(0)} \in C^\infty(M) \otimes U(n)$ , (2.38) has solutions which are determined recursively at each order of  $\hbar$  [27].

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<sup>2</sup>From several view points, matrix models and its related topics studied in [22, 23, 24] are useful for understanding the gauge theory constructed in this subsection.



In noncommutative Kähler manifolds, the ordinary exterior derivative  $d$  does not play an essential role, since the Leibniz rule is failed;  $d(f * g) \neq df * g + f * dg$ . To construct a covariant derivative of a gauge theory, we should adopt some derivations (operators which satisfy the Leibniz rule) instead of  $d$ . In particular, inner derivations are given by commutators of the star product. The space of inner derivations is infinite dimensional. Hence, if the whole space of inner derivations is used to construct gauge theories, the infinite number of gauge fields would be introduced. (See for example [10, 11] .) In this article, we consider deformation quantization of a homogeneous Kähler manifold  $\mathcal{G}/\mathcal{H}$  and choose a subalgebra of the Lie algebra of inner derivations. Then, we construct a deformation quantization of gauge theories on  $\mathcal{G}/\mathcal{H}$  whose covariant derivatives are derived from inner derivations corresponding to the Killing vector fields.

In a homogeneous Kähler manifold  $\mathcal{G}/\mathcal{H}$ , there are the holomorphic Killing vector fields  $\mathcal{L}_a = \zeta_a^i(z)\partial_i + \bar{\zeta}_a^{\bar{i}}(\bar{z})\partial_{\bar{i}}$  corresponding to the Lie algebra of the isometry group  $\mathcal{G}$ ,

$$[\mathcal{L}_a, \mathcal{L}_b] = if_{abc}\mathcal{L}_c, \quad (2.39)$$

where  $a$  is an index of the Lie algebra of  $\mathcal{G}$  and  $f_{abc}$  is its structure constant. There exists the Killing potential  $P_a$  corresponding to  $\mathcal{L}_a$ ,  $\mathcal{L}_a = \{P_a, \cdot\}$ . As stated in the previous section, the Killing vector  $\mathcal{L}_a$  can be described by  $*$ -commutator and satisfy the Leibniz rule,

$$\mathcal{L}_a = -\frac{i}{\hbar}[P_a, \cdot]_*, \quad (2.40)$$

$$\mathcal{L}_a(f * g) = (\mathcal{L}_a f) * g + f * (\mathcal{L}_a g). \quad (2.41)$$

The Killing vectors are normalized here as

$$\eta^{ab}\zeta_a^i\bar{\zeta}_b^{\bar{j}} = g^{i\bar{j}}, \quad \eta^{ab}\zeta_a^i\zeta_b^j = 0, \quad \eta^{ab}\bar{\zeta}_a^{\bar{i}}\bar{\zeta}_b^{\bar{j}} = 0, \quad (2.42)$$

where  $\eta^{ab}$  is the inverse of the Killing form of the Lie algebra of  $\mathcal{G}$ . We introduce gauge fields corresponding to  $\mathcal{L}_a$  in the following.

Let us consider a commutative homogeneous Kähler manifold  $M = \mathcal{G}/\mathcal{H}$ . We denote the indices of  $TM$  as  $\mu = 1, 2, \dots, 2N$  for combining the holomorphic and anti-holomorphic indices. We define  $\mathcal{A}_a^{(0)}$  as

$$\mathcal{A}_a^{(0)} = \zeta_a^\mu A_\mu = \zeta_a^i A_i + \bar{\zeta}_a^{\bar{i}} A_{\bar{i}}, \quad (2.43)$$

where  $A_i$  and  $A_{\bar{i}}$  are gauge fields on  $M$ . Its curvature is defined as

$$\mathcal{F}_{ab}^{(0)} := \mathcal{L}_a \mathcal{A}_b^{(0)} - \mathcal{L}_b \mathcal{A}_a^{(0)} - i[\mathcal{A}_a^{(0)}, \mathcal{A}_b^{(0)}] - if_{abc}\mathcal{A}_c^{(0)}, \quad (2.44)$$

where  $[A, B] = AB - BA$ .  $\mathcal{F}_{ab}^{(0)}$  is related to the curvature of  $A_\mu$ ,  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ , as

$$\mathcal{F}_{ab}^{(0)} = \zeta_a^\mu \zeta_b^\nu F_{\mu\nu}. \quad (2.45)$$

By using (2.42), it is shown that

$$\eta^{ac} \eta^{bd} \mathcal{F}_{ab}^{(0)} \mathcal{F}_{cd}^{(0)} = g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}. \quad (2.46)$$

Now, we consider a noncommutative deformation of gauge theories. We define

$$\mathcal{A}_a := \sum_{k=0}^{\infty} \hbar^k \mathcal{A}_a^{(k)} \quad (2.47)$$

as a gauge field, and define its gauge transformation by

$$\mathcal{A}_a \rightarrow \mathcal{A}'_a = iU^{-1} * \mathcal{L}_a U + U^{-1} * \mathcal{A}_a * U. \quad (2.48)$$

Let us define a curvature of  $\mathcal{A}_a$  by

$$\mathcal{F}_{ab} := \mathcal{L}_a \mathcal{A}_b - \mathcal{L}_b \mathcal{A}_a - i[\mathcal{A}_a, \mathcal{A}_b]_* - if_{abc} \mathcal{A}_c. \quad (2.49)$$

**Lemma 2.5.**  $\mathcal{F}_{ab}$  transforms covariantly:

$$\mathcal{F}_{ab} \rightarrow \mathcal{F}'_{ab} = U^{-1} * \mathcal{F}_{ab} * U. \quad (2.50)$$

*Proof.*

$$\mathcal{F}'_{ab} = \mathcal{L}_a \mathcal{A}'_b - \mathcal{L}_b \mathcal{A}'_a - i[\mathcal{A}'_a, \mathcal{A}'_b]_* - if_{abc} \mathcal{A}'_c. \quad (2.51)$$

Using (2.48) and

$$0 = \mathcal{L}_a(U^{-1} * U) = \mathcal{L}_a U^{-1} * U + U^{-1} * \mathcal{L}_a U$$

which is obtained from the Leibniz rule for  $\mathcal{L}_a$ , the right hand side of (2.51) is written as

$$U^{-1} * \mathcal{F}_{ab} * U + iU^{-1} * [\mathcal{L}_a, \mathcal{L}_b]U + f_{abc}U^{-1} * \mathcal{L}_c U.$$

Noting that  $[\mathcal{L}_a, \mathcal{L}_b] = if_{abc} \mathcal{L}_c$ , we have  $\mathcal{F}'_{ab} = U^{-1} * \mathcal{F}_{ab} * U$ .  $\square$

Using this lemma, we obtain the gauge invariant action.

**Theorem 2.6.** *A gauge invariant action for the gauge field is given by*

$$S_g := \int_{\mathcal{G}/\mathcal{H}} \mu_g \operatorname{tr} \left( \eta^{ac} \eta^{bd} \mathcal{F}_{ab} * \mathcal{F}_{cd} \right), \quad (2.52)$$

where  $\mu_g$  is a trace density.

*Proof.* The gauge invariance of the action is obtained by (2.50) and the cyclic symmetry of the trace density. The existence of trace density,  $\int_M f * g \mu_g = \int_M g * f \mu_g$ , is guaranteed in [20].  $\square$

Scalar fields are also introduced as similar to commutative case. As an example, let us consider a complex scalar field  $\phi = \sum_k \phi^{(k)} \hbar^k$  and its hermitian conjugate  $\phi^\dagger$  which transform as the fundamental representation of the gauge group,

$$\phi \rightarrow \phi' = U^{-1} * \phi, \quad \phi^\dagger \rightarrow \phi'^\dagger = \phi^\dagger * U. \quad (2.53)$$

A covariant derivative for this scalar field is defined by

$$\nabla_a \phi := \mathcal{L}_a \phi - i \mathcal{A}_a * \phi, \quad (2.54)$$

and then this transforms covariantly;

$$\nabla_a' \phi' = U^{-1} * \nabla_a \phi. \quad (2.55)$$

Therefore we obtain the gauge invariant action.

**Theorem 2.7.** *Let  $\phi$  be a fundamental representation complex scalar field and  $\phi^\dagger$  be a hermitian conjugate of  $\phi$  whose gauge transformations are given by (2.53). Then, the following action is gauge invariant.*

$$S_\phi = \int_{\mathcal{G}/\mathcal{H}} \mu_g \left\{ \eta^{ab} \nabla_a \phi^\dagger * \nabla_b \phi + V(\phi^\dagger * \phi) \right\}, \quad (2.56)$$

where  $V$  is a potential as a function of one variable.

### 3 Gauge theories in noncommutative $\mathbb{C}P^N$ and $\mathbb{C}H^N$

In this section, as examples of the deformed gauge theories defined in the previous section, we will construct noncommutative gauge theories on  $\mathbb{C}P^N$  and  $\mathbb{C}H^N$  by using deformation quantization with separation variables.

### 3.1 Deformation quantization with separation variables of $\mathbb{C}P^N$ and $\mathbb{C}H^N$

We recall the results for the deformation quantization with separation of variables for  $\mathbb{C}P^N$  and  $\mathbb{C}H^N$  [34].

In the inhomogeneous coordinates  $z^i$  ( $i = 1, 2, \dots, N$ ), the Kähler potential of  $\mathbb{C}P^N$  is given by

$$\Phi = \ln(1 + |z|^2), \quad (3.1)$$

where  $|z|^2 = \sum_{k=1}^N z^k \bar{z}^k$ . The metric  $(g_{i\bar{j}})$  is

$$ds^2 = 2g_{i\bar{j}} dz^i d\bar{z}^j, \quad (3.2)$$

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi = \frac{(1 + |z|^2) \delta_{ij} - z^j \bar{z}^i}{(1 + |z|^2)^2}, \quad (3.3)$$

and the inverse of the metric  $(g^{\bar{i}j})$  is

$$g^{\bar{i}j} = (1 + |z|^2) (\delta_{ij} + z^j \bar{z}^i). \quad (3.4)$$

Recall that the left star multiplication for a function  $f$ ,  $L_f$ , is written by using  $L_{\bar{z}^l}$ , (2.20). The explicit expression for  $L_{\bar{z}^l}$  on  $\mathbb{C}P^N$  is given by

$$\begin{aligned} L_{\bar{z}^l} &= \bar{z}^l + \hbar D^{\bar{l}} + \sum_{n=2}^{\infty} \hbar^n \sum_{m=2}^n a_m^{(n)} \partial_{\bar{j}_1} \Phi \cdots \partial_{\bar{j}_{m-1}} \Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}} \\ &= \bar{z}^l + \sum_{m=1}^{\infty} \alpha_m(\hbar) \partial_{\bar{j}_1} \Phi \cdots \partial_{\bar{j}_{m-1}} \Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}}, \end{aligned} \quad (3.5)$$

where

$$\alpha_m(t) = \sum_{n=m}^{\infty} t^n a_m^{(n)}, \quad (3.6)$$

$$\alpha_1(t) = 1, \quad (3.7)$$

$$\alpha_m(t) = t^m \prod_{n=1}^{m-1} \frac{1}{1 - nt} = \frac{\Gamma(1 - m + 1/t)}{\Gamma(1 + 1/t)}, \quad (m \geq 2). \quad (3.8)$$

The function  $\alpha_m(t)$  actually coincides with the generating function for the Stirling numbers of the second kind  $S(n, k)$ , and  $a_m^{(n)}$  is related to  $S(n, k)$  as

$$a_m^{(n)} = S(n - 1, m - 1). \quad (3.9)$$

One of non-trivial star products is  $\bar{z}^i * z^j$ ,

$$\begin{aligned}\bar{z}^i * z^j &= \bar{z}^i z^j + \hbar \delta_{ij} (1 + |z|^2) {}_2F_1(1, 1; 1 - 1/\hbar; -|z|^2) \\ &\quad + \frac{\hbar}{1 - \hbar} \bar{z}^i z^j (1 + |z|^2) {}_2F_1(1, 2; 2 - 1/\hbar; -|z|^2),\end{aligned}\quad (3.10)$$

where  ${}_2F_1$  is the Gauss hypergeometric function.

For  $\mathbb{C}H^N$ , similar results are obtained. The Kähler potential and the metric are given by

$$\Phi = -\ln(1 - |z|^2), \quad (3.11)$$

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi = \frac{(1 - |z|^2) \delta_{ij} + \bar{z}^i z^j}{(1 - |z|^2)^2}, \quad (3.12)$$

$$g^{\bar{i}j} = (1 - |z|^2) (\delta_{ij} - \bar{z}^i z^j). \quad (3.13)$$

The operator  $L_{\bar{z}^l}$  is expanded as a power series of the noncommutative parameter  $\hbar$ , and has the following explicit representation,

$$\begin{aligned}L_{\bar{z}^l} &= \bar{z}^l + \hbar D^{\bar{l}} + \sum_{n=2}^{\infty} \hbar^n \sum_{m=2}^n (-1)^{n-1} a_m^{(n)} \partial_{\bar{j}_1} \Phi \cdots \partial_{\bar{j}_{m-1}} \Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}} \\ &= \bar{z}^l + \sum_{m=1}^{\infty} (-1)^{m-1} \beta_m(\hbar) \partial_{\bar{j}_1} \Phi \cdots \partial_{\bar{j}_{m-1}} \Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}},\end{aligned}\quad (3.14)$$

where

$$\beta_n(t) = (-1)^n \alpha_n(-t) = \frac{\Gamma(1/t)}{\Gamma(n + 1/t)}. \quad (3.15)$$

Then, one of non-trivial star products is  $\bar{z}^i * z^j$ ,

$$\begin{aligned}\bar{z}^i * z^j &= \bar{z}^i z^j + \hbar \delta_{ij} (1 - |z|^2) {}_2F_1(1, 1; 1 + 1/\hbar; |z|^2) \\ &\quad - \frac{\hbar}{1 + \hbar} \bar{z}^i z^j (1 - |z|^2) {}_2F_1(1, 2; 2 + 1/\hbar; |z|^2).\end{aligned}\quad (3.16)$$

We should comment on the relation between our previous results and those of preceding related works [1, 4, 16].

Balachandran *et al.* gave an explicit expression of  $*$  product on fuzzy  $\mathbb{C}P^n$ , using matrix regularization [1]. Their  $*$  product is expressed as a finite series. Though our  $*$  product is, in general, an infinite series in  $\hbar$ , it coincides with Balachandran's  $*$  product if we take  $\hbar = 1/L (L \in \mathbb{N})$ .

On the other hand, Bordemann *et al.* obtained a star product which has a similar form of an infinite series in the noncommutative parameter  $\hbar$  to our star

product [4]. In fact, their star product is shown to be equivalent to ours (see [34] section 3).

Also in [16], an explicit expression of a star product on fuzzy  $S^2$  is given as an infinite series in a noncommutative parameter, which coincides with our expression in the case of  $\mathbb{C}P^1$ .

### 3.2 Differentials on noncommutative $\mathbb{C}P^N$

In this section, we study differentials in a noncommutative  $\mathbb{C}P^N$  with the star product with separation of variables.

In  $\mathbb{C}P^N$ , the conditions  $D^i D^j P = 0$  and  $D^{\bar{i}} D^{\bar{j}} P = 0$  can be solved as

$$P = \frac{\alpha_i z^i + \bar{\alpha}_i \bar{z}^i + \beta_{ij} \bar{z}^i z^j}{1 + |z|^2}, \quad (3.17)$$

where  $\alpha_i$  and  $\beta_{ij} = \bar{\beta}_{ji}$  are complex parameters and  $|z|^2 = \sum_{i=1}^N z^i \bar{z}^i$ . The number of the real parameters is  $N^2 + 2N$  and these correspond to the  $SU(N+1)$  isometry transformations of  $\mathbb{C}P^N$ . In the following, we give concrete expressions of the Killing potentials corresponding to the generators of  $su(N+1)$ , the Lie algebra of  $SU(N+1)$ .

Homogeneous coordinates of  $\mathbb{C}P^N$

$$\{\xi^A | A = 0, 1, \dots, N\} = \{\xi^0, \xi^i | i = 1, 2, \dots, N\} \quad (3.18)$$

are related with inhomogeneous coordinates on the chart of  $\xi^0 \neq 0$ :

$$z^i = \frac{\xi^i}{\xi^0}, \quad \bar{z}^i = \frac{\bar{\xi}^i}{\bar{\xi}^0}, \quad (i = 1, 2, \dots, N). \quad (3.19)$$

Since Kähler potential is given by  $\Phi = \ln(1 + |z|^2)$ , the isometry of  $SU(N+1)$  with the homogeneous coordinates is given by

$$\delta \xi^A = i\theta^a (T_a)_{AB} \xi^B, \quad \delta \bar{\xi}^A = -i\theta^a \bar{\xi}^B (T_a)_{BA}, \quad (3.20)$$

and its Lie derivative is given by

$$\mathcal{L}_a = - (T_a)_{AB} \left( \xi^B \frac{\partial}{\partial \xi^A} - \bar{\xi}^A \frac{\partial}{\partial \bar{\xi}^B} \right), \quad (3.21)$$

$$[\mathcal{L}_a, \mathcal{L}_b] = i f_{abc} \mathcal{L}_c. \quad (3.22)$$

Here we introduce the generators  $(T_a)_{AB}$  of  $su(N+1)$  in the fundamental representation which satisfy the following relations,

$$[T_a, T_b] = if_{abc}T_c, \quad \text{Tr } T_a = 0, \quad (3.23)$$

$$\text{Tr } T_a T_b = \delta_{ab}, \quad (3.24)$$

$$(T_a)_{AB}(T_a)_{CD} = \delta_{AD}\delta_{BC} - \frac{1}{N+1}\delta_{AB}\delta_{CD}, \quad (3.25)$$

where  $f_{abc}$  is the structure constant of  $SU(N+1)$ ,  $a = 1, 2, \dots, N^2 + 2N$ , and  $A, B = 0, 1, \dots, N$ . Generators of the isometry  $SU(N+1)$  in the inhomogeneous coordinates are given as

$$\begin{aligned} \mathcal{L}_a = \zeta_a^i \partial_i + \bar{\zeta}_a^{\bar{i}} \partial_{\bar{i}} = & (T_a)_{00} (z^i \partial_i - \bar{z}^{\bar{i}} \partial_{\bar{i}}) + (T_a)_{0i} (z^i z^j \partial_j + \partial_{\bar{i}}) \\ & + (T_a)_{i0} (-\partial_i - \bar{z}^{\bar{j}} \partial_{\bar{j}}) + (T_a)_{ij} (-z^j \partial_i + \bar{z}^{\bar{j}} \partial_{\bar{j}}), \end{aligned} \quad (3.26)$$

and

$$\zeta_a^i := (T_a)_{00} z^i + (T_a)_{0j} z^j z^i - (T_a)_{i0} - (T_a)_{ij} z^j, \quad (3.27)$$

$$\bar{\zeta}_a^{\bar{i}} := -(T_a)_{00} \bar{z}^{\bar{i}} + (T_a)_{0i} - (T_a)_{j0} \bar{z}^{\bar{j}} \bar{z}^{\bar{i}} + (T_a)_{ji} \bar{z}^{\bar{j}}. \quad (3.28)$$

The quadratic forms of  $\zeta_a^i$  and  $\bar{\zeta}_a^{\bar{i}}$  become the metric,

$$\zeta_a^i \bar{\zeta}_a^{\bar{j}} = -(1 + |z|^2)(\delta_{ij} + z^i \bar{z}^{\bar{j}}) = -g^{i\bar{j}}, \quad (3.29)$$

$$\zeta_a^i \zeta_a^j = 0, \quad \bar{\zeta}_a^{\bar{i}} \bar{\zeta}_a^{\bar{j}} = 0. \quad (3.30)$$

As shown in section 2, the Killing vector fields can be represented by star commutators with the Killing potentials. In the case of  $\mathbb{C}P^N$ , using the concrete expressions of the star product in section 3.1,  $\mathcal{L}_a$  can be written as

$$\mathcal{L}_a f = -\frac{i}{\hbar} [P_a, f]_*. \quad (3.31)$$

$P_a$  are obtained as

$$\begin{aligned} P_a = & -i(T_a)_{AB} \left( \frac{\bar{\xi}^A \xi^B}{|\xi|^2} - \delta_{AB} \right) \\ = & i(T_a)_{00} (z^i \partial_i \Phi - 1) - i(T_a)_{0i} \partial_i \Phi - i(T_a)_{i0} \partial_i \Phi - i(T_a)_{ij} z^j \partial_i \Phi. \end{aligned} \quad (3.32)$$

Note that  $P_a$  is determined up to an additional constant. The Killing potentials  $P_a$  give a representation of the  $su(N+1)$  under the star commutator,

$$[P_a, P_b]_* = -\hbar f_{abc} P_c, \quad (3.33)$$

and the bilinear of  $P_a$  becomes a constant,

$$P_a * P_a = -N \left( \frac{1}{N+1} + \hbar \right). \quad (3.34)$$

The Killing potential  $P$  in (3.17) can be written in a linear combination of  $P_a$ .

The star commutators between  $P_a$  and a function  $f$  become the Lie derivative  $\mathcal{L}_a f$  of  $f$  corresponding to the generator  $T_a$ ,

$$\begin{aligned} -\frac{i}{\hbar}[P_a, f]_* &= \mathcal{L}_a f \\ &= [(T_a)_{00} (z^i \partial_i - \bar{z}^i \partial_{\bar{i}}) + (T_a)_{0i} (z^i z^j \partial_j + \partial_{\bar{i}}) \\ &\quad + (T_a)_{i0} (-\partial_i - \bar{z}^i \bar{z}^j \partial_{\bar{j}}) + (T_a)_{ij} (-z^j \partial_i + \bar{z}^i \partial_{\bar{j}})] f. \end{aligned} \quad (3.35)$$

As emphasized before, since the expression of the star product has the coordinate dependence, general vector fields do not satisfy the Leibniz rule. However, the Leibniz rule trivially holds for the Killing vector fields, because they are described as the star commutators,

$$\mathcal{L}_a(f * g) = -\frac{i}{\hbar}[P_a, f * g]_* = -\frac{i}{\hbar}[P_a, f]_* * g - \frac{i}{\hbar}f * [P_a, g] = (\mathcal{L}_a f) * g + f * (\mathcal{L}_a g). \quad (3.36)$$

### 3.3 Differentials on noncommutative $\mathbb{C}H^N$

As similar to the  $\mathbb{C}P^N$ , we give explicit expressions of inner derivations given by the Killing potential for  $\mathbb{C}H^N$ . The Killing potential satisfying  $D^i D^j P = 0$  and  $D^{\bar{i}} D^{\bar{j}} P = 0$  can be solved as

$$P = \frac{\alpha_i z^i + \bar{\alpha}_i \bar{z}^i + \beta_{ij} \bar{z}^i z^j}{1 - |z|^2}, \quad (3.37)$$

where  $\alpha_i$  and  $\beta_{ij} = \bar{\beta}_{ji}$  are complex parameters. In the following, we construct inner derivations corresponding to the isometry transformations.

We first summarize useful facts in the isometry of  $\mathbb{C}H^N$ . As homogeneous coordinates of  $\mathbb{C}H^N$  we denote

$$\{\zeta^A | A = 0, 1, \dots, N\} = \{\zeta^0, \zeta^i | i = 1, 2, \dots, N\}, \quad (3.38)$$

and their relation between with inhomogeneous coordinates on the chart  $\zeta^0 \neq 0$  are given by

$$z^i = \frac{\zeta^i}{\zeta^0}, \quad \bar{z}^i = \frac{\bar{\zeta}^i}{\bar{\zeta}^0}, \quad (i = 1, 2, \dots, N). \quad (3.39)$$



Since the Kähler potential is given by  $\Phi = -\ln(1 - |z|^2)$ , there is an  $SU(1, N)$  isometry. Let us summarize the notations of  $SU(1, N)$ .  $SU(1, N)$  transformations preserve

$$|\xi|^2 = \eta_{AB} \bar{\xi}^A \xi^B, \quad (3.40)$$

where the metric is defined by  $(\eta_{AB}) = \text{diag.}(1, \overbrace{-1, \dots, -1}^N)$ . In other words,  $SU(1, N)$  is defined as

$$U \in SU(1, N) \iff U^\dagger \eta U = \eta, \quad \det U = 1. \quad (3.41)$$

The Lie algebra  $su(1, N)$  is defined by

$$A \in su(1, N) \iff U = e^A \in SU(1, N) \iff \eta A^\dagger \eta = -A, \quad \text{Tr} A = 0. \quad (3.42)$$

As a basis, we choose  $(N+1) \times (N+1)$  matrices  $T_a$  ( $a = 1, 2, \dots, N^2 + 2N$ ) which satisfy the following relations,

$$\text{Tr} T_a = 0, \quad (3.43)$$

$$\begin{aligned} (T_a^\dagger)_{00} &= -(T_a)_{00}, \quad (T_a^\dagger)_{ij} = -(T_a)_{ij}, \\ (T_a^\dagger)_{0i} &= (T_a)_{0i}, \quad (T_a^\dagger)_{i0} = (T_a)_{i0}, \end{aligned} \quad (3.44)$$

$$\text{Tr} T_a T_b = h_{ab}, \quad (h_{ab}) = \text{diag.}(\overbrace{-1, \dots, -1}^{N^2}, \overbrace{1, \dots, 1}^{2N}), \quad (3.45)$$

$$T_a^\dagger = h_{ab} T_b, \quad (3.46)$$

$$[T_a, T_b] = f_{abc} T_c, \quad (f_{abc} \in \mathbb{R}), \quad (3.47)$$

$$h_{ab} (T_a)_{AB} (T_b)_{CD} = \delta_{AD} \delta_{BC} - \frac{1}{N+1} \delta_{AB} \delta_{CD}. \quad (3.48)$$

More explicit form of a basis is given in the appendix B. Using these notations, transformations and generators of the isometry  $SU(1, N)$  in the homogeneous coordinates are obtained as

$$\delta \xi^A = \theta^a (T_a)_{AB} \xi^B, \quad \delta \bar{\xi}^A = \theta^a \bar{\xi}^B (T_a^\dagger)_{BA}, \quad (3.49)$$

$$\mathcal{L}_a = -(T_a)_{AB} \xi^B \frac{\partial}{\partial \xi^A} - (T_a^\dagger)_{AB} \bar{\xi}^A \frac{\partial}{\partial \bar{\xi}^B}, \quad (3.50)$$

$$[\mathcal{L}_a, \mathcal{L}_b] = f_{abc} \mathcal{L}_c. \quad (3.51)$$

The generators of the isometry  $SU(1, N)$  in the inhomogeneous coordinates are

$$\begin{aligned} \mathcal{L}_a &= \zeta_a^i \partial_i + \bar{\zeta}_a^{\bar{i}} \partial_{\bar{i}} = (T_a)_{00} (z^i \partial_i - \bar{z}^{\bar{i}} \partial_{\bar{i}}) + (T_a)_{0i} (z^i z^j \partial_j - \partial_{\bar{i}}) \\ &\quad + (T_a)_{i0} (-\partial_i + \bar{z}^{\bar{j}} \partial_{\bar{j}}) + (T_a)_{ij} (-z^j \partial_i + \bar{z}^{\bar{j}} \partial_{\bar{j}}), \end{aligned} \quad (3.52)$$

and

$$\zeta_a^i := (T_a)_{00}z^i + (T_a)_{0j}z^jz^i - (T_a)_{i0} - (T_a)_{ij}z^j, \quad (3.53)$$

$$\bar{\zeta}_a^{\bar{i}} := -(T_a)_{00}\bar{z}^i - (T_a)_{0i} + (T_a)_{j0}\bar{z}^j\bar{z}^i + (T_a)_{ji}\bar{z}^j. \quad (3.54)$$

The quadratic forms of  $\zeta_a^i$  and  $\bar{\zeta}_a^{\bar{i}}$  become the metric,

$$\zeta_a^i \bar{\zeta}_b^{\bar{j}} h_{ab} = (1 - |z|^2)(\delta_{ij} - z^i \bar{z}^j) = g^{i\bar{j}}, \quad (3.55)$$

$$\zeta_a^i \zeta_b^j h_{ab} = 0, \quad \bar{\zeta}_a^{\bar{i}} \bar{\zeta}_b^{\bar{j}} h_{ab} = 0. \quad (3.56)$$

As we found in general case (or similar to the case of  $\mathbb{CP}^N$ ), the Killing vector fields are written by commutators of the Killing potentials,

$$\mathcal{L}_a f = -\frac{i}{\hbar} [P_a, f]_*, \quad (3.57)$$

and the  $P_a$  are given by

$$\begin{aligned} P_a &= i(T_a)_{AB} \left( \frac{\eta_{AC} \bar{\xi}^C \xi^B}{|\xi|^2} - \delta_{AB} \right) \\ &= i(T_a)_{00} (z^i \partial_i \Phi + 1) + i(T_a)_{0i} \partial_i \Phi - i(T_a)_{i0} \partial_i \Phi - i(T_a)_{ij} z^j \partial_i \Phi. \end{aligned} \quad (3.58)$$

Note that  $P_a$  is determined up to an additional constant. The following formula is also obtained as similar to  $\mathbb{CP}^N$ :

$$P_a * P_b h_{ab} = -N \left( \frac{N}{N+1} - \hbar \right). \quad (3.59)$$

### 3.4 Cyclic property of integration and actions of gauge theories

In this section, we first show the cyclic property of integration, explicitly.

**Theorem 3.1.** *Let  $M$  be  $\mathbb{CP}^N$  or  $\mathbb{CH}^N$ , and let  $F$  and  $G$  be arbitrary compact supported bounded smooth functions on  $M$ . Then, the Riemannian volume form is a trace density with respect to the star products with separation of variables, namely we have*

$$\int_M F * G \sqrt{g} dz^1 \cdots dz^N d\bar{z}^1 \cdots d\bar{z}^N = \int_M G * F \sqrt{g} dz^1 \cdots dz^N d\bar{z}^1 \cdots d\bar{z}^N. \quad (3.60)$$

Note that the star products can be written by using the Levi-Civita connection  $\nabla_i$  and  $\nabla_{\bar{i}}$  as

$$F * G = FG + \sum_{n=1}^{\infty} c_n(\hbar) g^{\bar{i}_1 j_1} \dots g^{\bar{i}_n j_n} (\nabla_{\bar{i}_1} \dots \nabla_{\bar{i}_n} F) (\nabla_{j_1} \dots \nabla_{j_n} G), \quad (3.61)$$

where  $c_n(\hbar) = \alpha_n(\hbar)/n!$  for  $\mathbb{C}P^N$  and  $c_n(\hbar) = \beta_n(\hbar)/n!$  for  $\mathbb{C}H^N$  (see [34])<sup>3</sup>. We use the following relations which hold for the Levi-Civita connections and the Riemannian curvature tensor on  $\mathbb{C}P^N$  and  $\mathbb{C}H^N$  ([25] p169):

$$[\nabla_i, \nabla_j] = 0, \quad [\nabla_{\bar{i}}, \nabla_{\bar{j}}] = 0, \quad (3.62)$$

$$[\nabla_i, \nabla_{\bar{j}}] v_k = R_{i\bar{j}k}{}^l v_l, \quad [\nabla_i, \nabla_{\bar{j}}] v_{\bar{k}} = R_{i\bar{j}\bar{k}}{}^{\bar{l}} v_{\bar{l}}, \quad (3.63)$$

$$R_{i\bar{j}k}{}^l = -c(\delta_{kl} g_{i\bar{j}} + \delta_{il} g_{k\bar{j}}), \quad R_{i\bar{j}\bar{k}}{}^{\bar{l}} = -c(\delta_{kl} g_{j\bar{i}} + \delta_{il} g_{j\bar{k}}), \quad (3.64)$$

$$\nabla_m R_{i\bar{j}k}{}^l = \nabla_{\bar{m}} R_{i\bar{j}k}{}^l = \nabla_m R_{i\bar{j}\bar{k}}{}^{\bar{l}} = \nabla_{\bar{m}} R_{i\bar{j}\bar{k}}{}^{\bar{l}} = 0. \quad (3.65)$$

Here  $c = 1$  and  $c = -1$  are for  $\mathbb{C}P^N$  and  $\mathbb{C}H^N$ , respectively. To prove the theorem 3.1, we use the following lemma.

**Lemma 3.2.** *For the arbitrary  $C^\infty$  function  $G$  on  $M$ ,*

$$\nabla_{\bar{i}_1} \dots \nabla_{\bar{i}_n} \nabla_{j_1} \dots \nabla_{j_n} G = \nabla_{j_1} \dots \nabla_{j_n} \nabla_{\bar{i}_1} \dots \nabla_{\bar{i}_n} G. \quad (3.66)$$

The proof of this lemma is given in the appendix A.

Theorem 3.1 can be shown easily by using this lemma.

*Proof.*

$$\begin{aligned} \int d\mu F * G &= \int d\mu \left[ FG + \sum_{n=1}^{\infty} c_n(\hbar) g^{\bar{i}_1 j_1} \dots g^{\bar{i}_n j_n} (\nabla_{\bar{i}_1} \dots \nabla_{\bar{i}_n} F) (\nabla_{j_1} \dots \nabla_{j_n} G) \right] \\ &= \int d\mu \left[ GF + \sum_{n=1}^{\infty} (-1)^n c_n(\hbar) g^{\bar{i}_1 j_1} \dots g^{\bar{i}_n j_n} F (\nabla_{\bar{i}_1} \dots \nabla_{\bar{i}_n} \nabla_{j_1} \dots \nabla_{j_n} G) \right] \\ &= \int d\mu \left[ GF + \sum_{n=1}^{\infty} (-1)^n c_n(\hbar) g^{\bar{i}_1 j_1} \dots g^{\bar{i}_n j_n} F (\nabla_{j_1} \dots \nabla_{j_n} \nabla_{\bar{i}_1} \dots \nabla_{\bar{i}_n} G) \right] \\ &= \int d\mu \left[ GF + \sum_{n=1}^{\infty} c_n(\hbar) g^{\bar{i}_1 j_1} \dots g^{\bar{i}_n j_n} (\nabla_{\bar{i}_1} \dots \nabla_{\bar{i}_n} G) (\nabla_{j_1} \dots \nabla_{j_n} F) \right] \\ &= \int d\mu G * F \end{aligned} \quad (3.67)$$

where  $d\mu$  is the volume form on  $\mathbb{C}P^N$  or  $\mathbb{C}H^N$  written in (3.60).  $\square$

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<sup>3</sup> Do not confuse the Levi-Civita connection  $\nabla_i$  with the gauge covariant derivative (2.54).

This result is possible to be extended to functions of formal power series of bounded smooth functions with compact supports.

In section 2, we constructed a gauge theory on general noncommutative homogeneous Kähler manifolds. In particular, we consider gauge theory on the noncommutative  $\mathbb{C}P^N \approx \text{SU}(N+1)/\text{S}(\text{U}(1) \times \text{U}(N))$  and  $\mathbb{C}H^N \approx \text{SU}(1, N)/\text{S}(\text{U}(1) \times \text{U}(N))$  with separation of variables. In the previous section, the derivations for functions on noncommutative Kähler manifolds with isometry, and concrete expressions of the derivations for  $\mathbb{C}P^N$  and  $\mathbb{C}H^N$  are constructed. Using them, a gauge theory with gauge group  $G$  on the coset space is constructed. In addition, trace density is given by usual volume density as we see in this section. Then the action for the gauge fields is given by

$$S_g := \int_{\mathbb{C}P^N} \sqrt{g} dz^1 \cdots dz^N d\bar{z}^1 \cdots d\bar{z}^N \text{tr} \left( \mathcal{F}_{ab} * \mathcal{F}_{cd} \eta^{ac} \eta^{bd} \right), \quad (3.68)$$

where  $\text{tr}$  is trace for gauge group  $G$ . The gauge invariance of the action is guaranteed by (2.50) and the cyclic symmetry. The action for the scalar field are same as (2.56);

$$S_\phi = \int_M \sqrt{g} dz^1 \cdots dz^N d\bar{z}^1 \cdots d\bar{z}^N \{ \nabla_a \phi^\dagger * \nabla_b \phi \eta^{ab} + V(\phi^\dagger * \phi) \}. \quad (3.69)$$

## 4 Conclusions

As we showed in Section 2.2, vector fields on noncommutative spaces are not derivations in general. We proved that the vector fields satisfying the Leibniz rule are characterized as the Killing vector fields, and their operators are given by star commutators of the Killing potentials on general noncommutative Kähler manifolds for the deformation quantization with separation of variables. We focused on a gauge theory which has derivations given by order one differential operators, by only considering inner derivations which possess vector fields expressions on general homogeneous Kähler manifolds. As examples, we constructed explicit expressions for these inner derivations on  $\mathbb{C}P^N$  and  $\mathbb{C}H^N$ . For our deformation quantization, we directly proved that integrations of  $*$ -products of functions with the volume form of the Kähler metric of  $\mathbb{C}P^N$  and  $\mathbb{C}H^N$  have a cyclic property. We then constructed an action functional having gauge symmetry on these manifolds.

We note that the action functionals given in this article are gauge invariants not only for noncommutative homogeneous Kähler manifolds but also for the isometry groups of general noncommutative Kähler manifolds. In this sense, gauge theories on general noncommutative Kähler manifolds are constructed in this article. However, the relation between the usual action of gauge fields (2.46) and the normalization (2.42) is obtained only for noncommutative homogeneous Kähler manifolds.

In other words, the correspondence between the gauge theories on a commutative space and the noncommutative space is clear, and it is possible to interpret the noncommutative gauge theory as a deformation of the commutative gauge theory for noncommutative homogeneous Kähler manifolds.

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## A The proof of the lemma 3.2

We give the proof of the lemma 3.2,

$$\nabla_{\bar{i}_1} \cdots \nabla_{\bar{i}_n} \nabla_{j_1} \cdots \nabla_{j_n} G = \nabla_{j_1} \cdots \nabla_{j_n} \nabla_{\bar{i}_1} \cdots \nabla_{\bar{i}_n} G. \quad (\text{A.1})$$

*Proof.* When  $n = 1$ , trivially

$$\nabla_{\bar{i}} \nabla_j G = \nabla_j \nabla_{\bar{i}} G. \quad (\text{A.2})$$

Assume  $\nabla_{\bar{i}_1} \cdots \nabla_{\bar{i}_{n-1}} \nabla_{j_1} \cdots \nabla_{j_{n-1}} G = \nabla_{j_1} \cdots \nabla_{j_{n-1}} \nabla_{\bar{i}_1} \cdots \nabla_{\bar{i}_{n-1}} G$ . We here use the following notation for simplicity,

$$[k, l] \equiv g_{\bar{i}_k j_l} \nabla_{\bar{i}_1} \cdots \hat{\nabla}_{\bar{i}_k} \cdots \nabla_{\bar{i}_n} \nabla_{j_1} \cdots \hat{\nabla}_{j_l} \cdots \nabla_{j_n} G, \quad (\text{A.3})$$

where “ $\hat{A}$ ” means  $A$  is removed. Then,

$$\begin{aligned}
\nabla_{\bar{i}_1} \cdots \nabla_{\bar{i}_n} \nabla_{j_1} \cdots \nabla_{j_n} G &= \nabla_{\bar{i}_1} \nabla_{j_1} (\nabla_{\bar{i}_2} \cdots \nabla_{\bar{i}_n} \nabla_{j_2} \cdots \nabla_{j_n} G) \\
&\quad + \sum_{k=2}^n \nabla_{\bar{i}_1} \cdots \nabla_{\bar{i}_{k-1}} [\nabla_{\bar{i}_k}, \nabla_{j_1}] \nabla_{\bar{i}_{k+1}} \cdots \nabla_{\bar{i}_n} \nabla_{j_2} \cdots \nabla_{j_n} G \\
&= \nabla_{\bar{i}_1} \nabla_{j_1} (\nabla_{j_2} \cdots \nabla_{j_n} \nabla_{\bar{i}_2} \cdots \nabla_{\bar{i}_n}) G \\
&\quad + \sum_{k=2}^n \nabla_{\bar{i}_1} \cdots \nabla_{\bar{i}_{k-1}} \left[ \sum_{l=k+1}^n R_{\bar{i}_k j_1 \bar{i}_l} \bar{p} \nabla_{\bar{i}_{k+1}} \cdots \nabla_{\bar{i}_n}^{(l)} \nabla_{j_2} \cdots \nabla_{j_n} G \right. \\
&\quad \left. + \sum_{l=2}^n R_{\bar{i}_k j_1 j_l} {}^p \nabla_{\bar{i}_{k+1}} \cdots \nabla_{\bar{i}_n} \nabla_{j_2} \cdots \nabla_p \cdots \nabla_{j_n} G \right] \\
&= \nabla_{\bar{i}_1} \nabla_{j_1} (\nabla_{j_2} \cdots \nabla_{j_n} \nabla_{\bar{i}_2} \cdots \nabla_{\bar{i}_n}) G \\
&\quad + c \sum_{k=2}^n \left[ - \sum_{l=k+1}^n ([k, 1] + [l, 1]) + \sum_{l=2}^n ([k, 1] + [k, l]) \right] \\
&= \nabla_{\bar{i}_1} \nabla_{j_1} (\nabla_{j_2} \cdots \nabla_{j_n} \nabla_{\bar{i}_2} \cdots \nabla_{\bar{i}_n}) G \\
&\quad + c \sum_{k=2}^n (k-1)[k, 1] - c \sum_{k=2}^{n-1} \sum_{l=k+1}^n [l, 1] + c \sum_{k=2}^n \sum_{l=2}^n [k, l] \\
&= \nabla_{\bar{i}_1} \nabla_{j_1} (\nabla_{j_2} \cdots \nabla_{j_n} \nabla_{\bar{i}_2} \cdots \nabla_{\bar{i}_n}) G \\
&\quad + c \sum_{k=2}^n (k-1)[k, 1] - c \sum_{l=3}^n \sum_{k=2}^{l-1} [l, 1] + c \sum_{k=2}^n \sum_{l=2}^n [k, l] \\
&= \nabla_{\bar{i}_1} \nabla_{j_1} (\nabla_{j_2} \cdots \nabla_{j_n} \nabla_{\bar{i}_2} \cdots \nabla_{\bar{i}_n}) G \\
&\quad + c \sum_{k=2}^n (k-1)[k, 1] - c \sum_{l=3}^n (l-2)[l, 1] + c \sum_{k=2}^n \sum_{l=2}^n [k, l] \\
&= \nabla_{\bar{i}_1} \nabla_{j_1} (\nabla_{j_2} \cdots \nabla_{j_n} \nabla_{\bar{i}_2} \cdots \nabla_{\bar{i}_n}) G + c \sum_{k=2}^n [k, 1] + c \sum_{k=2}^n \sum_{l=2}^n [k, l].
\end{aligned} \tag{A.4}$$

Next, the first term in the last expression,  $\nabla_{\bar{i}_1} \nabla_{j_1} (\nabla_{j_2} \cdots \nabla_{j_n} \nabla_{\bar{i}_2} \cdots \nabla_{\bar{i}_n}) G$  be-

comes

$$\begin{aligned}
\nabla_{\bar{i}_1} \nabla_{j_1} \nabla_{j_2} \cdots \nabla_{j_n} \nabla_{\bar{i}_2} \cdots \nabla_{\bar{i}_n} G &= \nabla_{j_1} \cdots \nabla_{j_n} \nabla_{\bar{i}_1} \cdots \nabla_{\bar{i}_n} G \\
&+ \sum_{k=1}^n \nabla_{j_1} \cdots \nabla_{j_{k-1}} [\nabla_{\bar{i}_1}, \nabla_{j_k}] \nabla_{j_{k+1}} \cdots \nabla_{j_n} \nabla_{\bar{i}_2} \cdots \nabla_{\bar{i}_n} G \\
&= \nabla_{j_1} \cdots \nabla_{j_n} \nabla_{\bar{i}_1} \cdots \nabla_{\bar{i}_n} G \\
&+ \sum_{k=1}^n \nabla_{j_1} \cdots \nabla_{j_{k-1}} \left[ \sum_{l=k+1}^n R_{\bar{i}_1 j_k j_l} {}^p \nabla_{j_{k+1}} \cdots \nabla_p^{(l)} \cdots \nabla_{j_n} \nabla_{\bar{i}_2} \cdots \nabla_{\bar{i}_n} G \right. \\
&\quad \left. + \sum_{l=2}^n R_{\bar{i}_1 j_k \bar{i}_l} {}^{\bar{p}} \nabla_{j_{k+1}} \cdots \nabla_{j_n} \nabla_{\bar{i}_2} \cdots \nabla_{\bar{p}}^{(l)} \cdots \nabla_{\bar{i}_n} G \right] \\
&= \nabla_{j_1} \cdots \nabla_{j_n} \nabla_{\bar{i}_1} \cdots \nabla_{\bar{i}_n} G \\
&+ c \sum_{k=1}^n \left[ \sum_{l=k+1}^n ([1, k] + [1, l]) - \sum_{l=2}^n ([1, k] + [l, k]) \right] \\
&= \nabla_{j_1} \cdots \nabla_{j_n} \nabla_{\bar{i}_1} \cdots \nabla_{\bar{i}_n} G \\
&- c \sum_{k=2}^n (k-1)[1, k] + c \sum_{k=1}^n \sum_{l=k+1}^n [1, l] - c \sum_{k=1}^n \sum_{l=2}^n [l, k] \\
&= \nabla_{j_1} \cdots \nabla_{j_n} \nabla_{\bar{i}_1} \cdots \nabla_{\bar{i}_n} G - c \sum_{l=2}^n [l, 1] - c \sum_{k=2}^n \sum_{l=2}^n [l, k].
\end{aligned} \tag{A.5}$$

This completes the proof for the lemma.  $\square$

## B A basis of $su(1, N)$

A concrete basis of  $su(1, N)$ ,  $T_a, (a = 1, 2, \cdots, (N+1)^2 - 1)$  is given as follows;

$$\{T_a\} = \{I_{ij}, J_{ij}, H_k, I_{0i}, J_{0i}\}, \tag{B.1}$$

where  $i, j, k = 1, 2, \dots, N$  and  $i < j$  in  $I_{ij}, J_{ij}$ .

$$I_{ij} = \frac{1}{\sqrt{2}}(E_{ij} - E_{ji}), \quad (\text{B.2})$$

$$J_{ij} = \frac{i}{\sqrt{2}}(E_{ij} + E_{ji}), \quad (\text{B.3})$$

$$H_k = \frac{i}{\sqrt{k(k+1)}} \left( \sum_{i=1}^k E_{ii} - kE_{k+1,k+1} \right), \quad (E_{N+1,N+1} = E_{00}), \quad (\text{B.4})$$

$$I_{0i} = \frac{1}{\sqrt{2}}(E_{i0} + E_{0i}), \quad (\text{B.5})$$

$$J_{0i} = \frac{i}{\sqrt{2}}(E_{i0} - E_{0i}), \quad (\text{B.6})$$

where  $(E_{AB})_{CD} = \delta_{AC}\delta_{BD}$  and  $A, B, C, D = 0, 1, \dots, N$ .  $I_{ij}, J_{ij}, H_k$  are anti-hermitian and  $I_{0i}, J_{0i}$  are hermitian.

## References

- [1] A. P. Balachandran, B. P. Dolan, J. -H. Lee, X. Martin and D. O'Connor, "Fuzzy complex projective spaces and their star products," J. Geom. Phys. **43**, 184 (2002) [hep-th/0107099].
- [2] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, "M theory as a matrix model: A Conjecture," Phys. Rev. D **55**, 5112 (1997) [hep-th/9610043].
- [3] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, "Deformation Theory And Quantization. 1. Deformations Of Symplectic Structures," Annals Phys. **111** (1978) 61.  
F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, "Deformation Theory And Quantization. 2. Physical Applications," Annals Phys. **111** (1978) 111.
- [4] M. Bordemann, M. Brischle, C. Emmrich, S. Waldmann, "Phase Space Reduction for Star-Products: An Explicit Construction for  $\mathbb{C}P^n$ ," Lett. Math. Phys. **36** (1996), 357.
- [5] M. Cahen, S. Gutt, J. Rawnsley, "Quantization of Kahler manifolds, II," Am. Math. Soc. Transl. **337**, 73 (1993).
- [6] M. Cahen, S. Gutt, J. Rawnsley, "Quantization of Kahler manifolds, IV," Lett. Math. Phys. **34**, 159 (1995).



- [7] U. Carow-Watamura and S. Watamura, “Noncommutative geometry and gauge theory on fuzzy sphere,” *Commun. Math. Phys.* **212**, 395 (2000) [hep-th/9801195].
- [8] M. R. Douglas and N. A. Nekrasov, (2001) [hep-th/0106048]. citations counted in INSPIRE as of 03 Mar 2014
- [9] M. De Wilde, P. B. A. Lecomte, “Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds,” *Lett. Math. Phys.* **7**, 487 (1983).
- [10] M. Dubois-Violette, J. Madore and R. Kerner, “Classical Bosons in a Noncommutative Geometry,” *Class. Quant. Grav.* **6**, 1709 (1989). citations counted in INSPIRE as of 22 Nov 2013
- [11] M. Dubois-Violette, “Lectures on graded differential algebras and noncommutative geometry,” math/9912017 [math-qa].
- [12] B. Fedosov, “A simple geometrical construction of deformation quantization,” *J. Differential Geom.* **40**, 213 (1994).
- [13] D. Z. Freedman and A. Van Proeyen, “Supergravity,” Cambridge, UK: Cambridge Univ. Pr. (2012) 607 p
- [14] R. Gopakumar, S. Minwalla and A. Strominger, “Noncommutative solitons,” *JHEP* **0005**, 020 (2000) [hep-th/0003160].
- [15] H. Grosse and H. Steinacker, “Finite gauge theory on fuzzy  $CP^{*2}$ ,” *Nucl. Phys. B* **707**, 145 (2005) [hep-th/0407089].
- [16] K. Hayasaka, R. Nakayama and Y. Takaya, “A New noncommutative product on the fuzzy two sphere corresponding to the unitary representation of  $SU(2)$  and the Seiberg-Witten map,” *Phys. Lett. B* **553**, 109 (2003) [hep-th/0209240].
- [17] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, “A Large  $N$  reduced model as superstring,” *Nucl. Phys. B* **498**, 467 (1997) [hep-th/9612115].
- [18] A. V. Karabegov, “On deformation quantization, on a Kahler manifold, associated to Berezin’s quantization,” *Funct. Anal. Appl.* **30**, 142 (1996).
- [19] A. V. Karabegov, “Deformation quantizations with separation of variables on a Kahler manifold,” *Commun. Math. Phys.* **180**, 745 (1996) [arXiv:hep-th/9508013].
- [20] A. V. Karabegov, “On the canonical normalization of a trace density of deformation quantization,” *Lett. Math. Phys.* **45** (1998) 217.
- [21] A. V. Karabegov, “An explicit formula for a star product with separation of variables,” [arXiv:1106.4112 [math.QA]].

- [22] H. Kawai, S. Shimasaki and A. Tsuchiya, “Large N reduction on group manifolds,” *Int. J. Mod. Phys. A* **25**, 3389 (2010) [arXiv:0912.1456 [hep-th]].
- [23] H. Kawai, S. Shimasaki and A. Tsuchiya, “Large N reduction on coset spaces,” *Phys. Rev. D* **81**, 085019 (2010) [arXiv:1002.2308 [hep-th]].
- [24] Y. Kitazawa, “Matrix models in homogeneous spaces,” *Nucl. Phys. B* **642**, 210 (2002) [hep-th/0207115].
- [25] S. Kobayashi and K. Nomizu, “Foundation of Differential Geometry, volume II,” John Wiley and Sons, Inc , 1969
- [26] M. Kontsevich, “Deformation quantization of Poisson manifolds, I,” *Lett. Math. Phys.* **66**, 157 (2003) [arXiv:q-alg/9709040].
- [27] Y. Maeda, A. Sako, *Are vortex numbers preserved?*, *J.Geom. Phys.* **58** (2008), 967-978 [math-ph/0612041](#).
- [28] S. Minwalla, M. Van Raamsdonk and N. Seiberg, “Noncommutative perturbative dynamics,” *JHEP* **0002**, 020 (2000) [hep-th/9912072].
- [29] C. Moreno, “\*-products on some Kähler manifolds”, *Lett. Math. Phys.* **11**, 361 (1986).
- [30] C. Moreno, “Invariant star products and representations of compact semisimple Lie groups,” *Lett. Math. Phys.* **12**, 217 (1986).
- [31] N. Nekrasov and A. S. Schwarz, “Instantons on noncommutative  $\mathbb{R}^{4|2}$  and (2,0) superconformal six-dimensional theory,” *Commun. Math. Phys.* **198**, 689 (1998) [hep-th/9802068].
- [32] H. Omori, Y. Maeda, and A. Yoshioka, “Weyl manifolds and deformation quantization,” *Adv. in Math.* **85**, 224 (1991).
- [33] A. Sako, *Recent developments in instantons in noncommutative  $\mathbb{R}^4$* , *Adv. Math. Phys.* **2010**(2010) , ID 270694, 28pp.
- [34] A. Sako, T. Suzuki and H. Umetsu, “Explicit Formulas for Noncommutative Deformations of  $CP^N$  and  $CH^N$ ,” *J. Math. Phys.* **53**, 073502 (2012) [arXiv:1204.4030 [math-ph]].
- [35] A. Sako, T. Suzuki and H. Umetsu, “Noncommutative  $CP^N$  and  $CH^N$  and their physics,” *J. Phys. Conf. Ser.* **442**, 012052 (2013).
- [36] N. Seiberg and E. Witten, “String theory and noncommutative geometry,” *JHEP* **9909**, 032 (1999) [hep-th/9908142].
- [37] R. J. Szabo, “Quantum field theory on noncommutative spaces,” *Phys. Rept.* **378**, 207 (2003) [hep-th/0109162].